# Application of Meshfree Methods in Predicting Solid Material Behavior

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**Abstract**

*Cracks propagations, fragmentations and large deformation of engineering solid materials are characterized by a continuous change in the geometry of the domain under consideration. Conventional methods such as finite element and finite difference method are usually used to analyze these types of problems with difficulties. These methods use meshes to model and analyze the material behavior. The analysis of large deformation and crack propagation of engineering problems by these methods may require the continuous remeshing of the domain in order to avoid the break down of the calculation due to excessive mesh distortion. In the analysis when even only a few meshes are needed, mesh generation can consume more time and effort compared to the construction and solution of the discrete set of equations. In this paper the essential boundary condition with the enforcing of penalty method is modified. A more recently introduced mesh-free method which provides an attractive solution to this kind of problems is presented. In this method, the domain is represented by a set of arbitrary distributed nodes and there is no need to use meshes or elements for field variable interpolation. The nodes remain constant while the geometry of the domain is changing, therefore saving time and computational effort in analyzing process. A few examples of application of the method in predicting the behavior of materials with large deformations are presented and its suitability is highlighted.*

**Keyword:** *Numerical Method, Meshless Methods, Element Free Galerkin Method.*

## 1- Introduction

The computational problems in engineering branch grow ever more interceding. Like crack propagation, fragmentation and large deformation in the simulation of manufacturing processes for solid and liquids. We need to model the large deformation and crack propagation properly with arbitrary paths. The analysis of these problems with conventional computational methods such as finite element and finite difference are not well proper. The analysis of large deformation problems by the method based on meshes may require the remeshing of the domain in each step the evolution. This strategy is proper for method based on meshes but introduce numerous difficulties. The continuous remeshing of the domain in each step of the evolution leads to degradation of accuracy and complexity in the computer program and the time-consuming mesh generation. Over the past three decades, many researchers have come to realize that so-called meshless methods can be developed that eliminate the meshes and their difficulties. Then the meshless methods there are only nodes. Although must be taken to meshes in at least parts of the some of meshless methods, often be treated without remeshing in shap function with minor costs in accuracy degradation. In this manner some of meshless methods basically no require meshes in background, such as Finite Point Method (FPM). Therefore the using of methods based on meshes to solve large classes of problems or three dimensional problems are very awkward. About thirty years ago until recently the many researchers have been developed type of meshless methods. T. P. Fries and H. G. Matthies [1] published a special issue on meshless in July 2004 that is classification and overview of meshfree methods. They introduced type of meshless methods such as Smooth Particle Hydrodynamics (SPH), Diffuse Element Method (DEM), Element Free Galerkin (EFG), Least Squares Meshfree Method (LSMM), Meshfree Local Petrov Galerkin (MLPG), Local Boundary Integral Equation (LBIE), Partition of Unity Methods (PUM), hp clouds, Natural Element Method (NEM), Meshless Finite Element Method (MFEM), Reproducing Kernel Element Method (RKEM). In this paper EFG method is used and for enforcing the essential boundary conditions is applied modified penalty method. The example is shown good results and calculated time is less than FDM[1].

**2- Element Free Galerkin Method (EFG)**

Belytschko et al. [2] modified the constructing shape function for Diffuse Element Method (DEM). They named it the Element Free Galerkin (EFG) method. The Moving Least Squares (MLS) approximation procedure related to construct shape function for EFG method. The MLS approximation $u^h(\mathbf{x})$ is defined in the form of:

$$u^h(\mathbf{x})=\sum_{j=1}^{m} p_j(\mathbf{x})a_j(\mathbf{x})\equiv\mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{1}$$

where $p_j(\mathbf{x})$ are monomials of basis function in the space coordinates

$$\mathbf{x}^T=[x,y,z] \tag{2}$$

In 1D space is provided by:

$$\mathbf{P}^T(\mathbf{x})=[1,x,x^2, \dots ,x^m] \tag{3}$$

and in 2D space:

$$\mathbf{P}^T(x,y)=[1,x,y,xy,x^2,y^2, \dots ,x^m,y^m] \tag{4}$$

and in 3D space, we have:

$$\mathbf{P}^T(x,y,z)=[1,x,y,z,xy,yz,zx,x^2,y^2,z^2, \dots ,x^m,y^m,z^m] \tag{5}$$

where m is the number of terms of monomials (polynomial basis).

[1] . Finite Difference Method.

$\boldsymbol{a}(\mathbf{x})$ is a vector of coefficients and is obtained at any point $\mathbf{x}$ by minimizing J.

$$J=\sum_{i=1}^{m} w(\mathbf{x}-\mathbf{x}_I)[\mathbf{p}^T(\mathbf{x}_I)\mathbf{a}(\mathbf{x})-u_I]^2 \qquad (6)$$

Where J is a function of weighted residual and constructed using the approximated values of the field function**.** The stationary of J with $\mathbf{a(x)}$:

$$\mathbf{A(x)a(x)=B(x)u}$$

(7)

or

$$\mathbf{a(x)=A^{-1}(x)B(x)u} \qquad (8)$$

where

$$\mathbf{A(x)}=\sum_{I=1}^{n} w_I(\mathbf{x})\mathbf{p}^T(\mathbf{x}_I)\mathbf{p}(\mathbf{x}_I) \qquad (9)$$

$$\mathbf{B(x)}=[w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1),\, w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2),\, \dots,\, w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)] \qquad (10)$$

$$\mathbf{u}=[u_1, u_2, \dots, u_n] \qquad (11)$$

Substituting the equation (8) into (1) leads to:

$$u^h(\mathbf{x})=\sum_{I=1}^{n}\sum_{j=1}^{m} p_j(\mathbf{x})(\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}))_{jI}u_I \qquad (12)$$

or

$$u^h(\mathbf{x})=\sum_{I=1}^{n} \Phi_I(\mathbf{x})\, u_I \qquad (13)$$

where the MLS shape function $\Phi_I(\mathbf{x})$ is defined by:

$$\Phi_I(\mathbf{x})= \sum_{j=1}^{m} p_j(\mathbf{x})(\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}))_{jI}\equiv\mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \qquad (14)$$

The partial derivative of $\Phi_I(\mathbf{x})$ can be obtained as follows:

$$\Phi_{I,i}= \sum_{j=1}^{m}\left\{p_{j,i}(\mathbf{A}^{-1}\mathbf{B})_{jI}+p_j\mathbf{A}^{-1}(\mathbf{B}_{,i}-A_{,i}\mathbf{A}^{-1}\mathbf{B})_{jI}\right\} \qquad (15)$$

The weight function $w_I(\mathbf{x})$ is positive and an important coefficient. In this paper it is:

$$w_I(\mathbf{x})=\begin{cases} 1-6*s^2+8*s^3-3*s^4 & \text{for} \quad : d_I<=r \\ 0 & \text{for} \quad : d_I>r \end{cases} \qquad (16)$$

where $s=d_I/r$ , $d_I=\|\mathbf{x}-\mathbf{x}_I\|$ , r=influence domain.

The weight function is large for $\mathbf{x}_I$ close to $\mathbf{x}$ and is small for $\mathbf{x}_I$ far from $\mathbf{x}$ and is zero for out of influence domain.

**3- Displacement, Strain and Stress**

The displacement of any point in the domainis obtain ed by:

$$u^h(\mathbf{x})=\boldsymbol{\Phi}(\mathbf{x})\mathbf{u} \quad (17)$$

where $\boldsymbol{\Phi}(\mathbf{x})$ is shape function and it is defined by:

$$\boldsymbol{\Phi}(\mathbf{x})= \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \quad (18)$$

The details of (18) were shown the previous section. $\mathbf{u}_a$ is displacement vector of nodes in the influence domain $\mathbf{x}$ .
The displacement vector of nodes is obtained from equilibrium equation. The equilibrium equation can be obtained from variational principles. The functional of the total potential energy of a material is given by:

$$\Pi=\Pi_b+ \Pi_f + \Pi_p \quad (19)$$

Where $\Pi_b$ is the elastic strain energy of block. $\Pi_f$ is the potential energy of the body force $f_e$ and $\Pi_p$ is the potential energy of the concentrated force p. They are given by:

$$\Pi_b=\sum_e t_e\iint_{\Omega e}\tfrac{1}{2}\varepsilon^T\mathbf{D}_b\, \varepsilon\, d_xd_y=\tfrac{1}{2}\sum_e\mathbf{u}_b{}^T(t_e \iint_{\Omega e} \mathbf{B}_b{}^T\, \mathbf{D}_b\mathbf{B}_b d_xd_y)\mathbf{u}_b \quad (20)$$

$$\Pi_f=-\sum_e \iint_{\Omega e}\mathbf{u}^T\mathbf{f}_e\, d_xd_y=-\sum_e\mathbf{u}_b{}^T(\iint_{\Omega e} \mathbf{N}^T\, \mathbf{f}_e\, d_xd_y) \quad (21)$$

$$\Pi_p=-\sum_m \mathbf{u}^T\mathbf{p}_m =-\sum_m\mathbf{u}_b{}^T(\mathbf{N}^T\, \mathbf{p}_m) \quad (22)$$

We can obtain equilibrium equation from the stationary of functional Π in (19). The equilibrium equation of solid materials is given by:

$$\mathbf{KU=P} \quad (23)$$

Where

$$\mathbf{K}=\sum_e t_e \iint_{\Omega e} \mathbf{B}_b{}^T\, \mathbf{D}_b\mathbf{B}_b d_xd_y \quad (24)$$

$$\mathbf{P}=\sum_e\iint_{\Omega e} \mathbf{N}^T\, \mathbf{f}_e\, d_xd_y +\sum_m \mathbf{N}^T\mathbf{p}_m \quad (25)$$

$$\mathbf{U}=[u_1,v_1,u_2,v_2,\ldots,u_n,v_n]^T \quad (26)$$

Where $t_e$ is the thickness of the material e, n is the total number of nodes in the problem domain, **U** is displacement vector related to total number of nodes in the problem domain. The displacement of any point into problem domain is obtained by:

$$\mathbf{u}=\mathbf{N}.\mathbf{u}_b \quad (27)$$

where

$$\mathbf{N}=\begin{bmatrix} \Phi_1(\mathbf{x}) , 0 , \Phi_2(\mathbf{x}) , 0 , \ldots , \Phi_n(\mathbf{x}) , 0 \\ \end{bmatrix} \quad (28)$$

$0 , \Phi_1(\mathbf{x}) , 0 , \Phi_2(\mathbf{x}) , 0 , \dots , \Phi_n(\mathbf{x})$

and $\mathbf{u}_b$ is the displacement vector of influence domain **x.** The Strain and Stress at any point **x** are given by:

$$\boldsymbol{\varepsilon}=\mathbf{B}_b\mathbf{u}_b \quad (29)$$

$$\boldsymbol{\sigma}=\mathbf{D}_b\mathbf{B}_b\mathbf{u}_b \quad (30)$$

where

$$\mathbf{B}_b=\begin{pmatrix} \Phi_{1,x}(\mathbf{x}) & ,0, & \Phi_{2,x}(\mathbf{x}), & 0,\dots, & \Phi_{n,x}(\mathbf{x}), & 0 \\ 0 & , \Phi_{1,y}(\mathbf{x}) & ,0 & , \Phi_{2,y}(\mathbf{x}) & ,0,\dots, & \Phi_{n,y}(\mathbf{x}) \\ \Phi_{1,y}(\mathbf{x}) & ,\Phi_{1,x}(\mathbf{x}) & , \dots, & & \Phi_{n,y}(\mathbf{x}) , & \Phi_{n,x}(\mathbf{x}) \end{pmatrix} \quad (31)$$

and

$$\mathbf{D}_b=(E/(1-\upsilon^2))\begin{pmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{pmatrix} \quad \text{for plain stress state} \quad (32)$$

$$\mathbf{D}_b=(E/(1-2\upsilon)(1+\upsilon))\begin{pmatrix} 1-\upsilon & \upsilon & 0 \\ \upsilon & 1-\upsilon & 0 \\ 0 & 0 & (1-2\,\upsilon)/2 \end{pmatrix} \quad \text{for plain strain state} \quad (33)$$

and

$$\boldsymbol{\varepsilon}=[\varepsilon_x,\varepsilon_y,\varepsilon_{xy}]^T \quad (34)$$

$$\boldsymbol{\sigma}=[\sigma_x, \sigma_y, \sigma_{xy}]^T \quad (35)$$

**4- The equilibrium equation with enforcing the essential boundary condition**

The nodal value of the interpolation function $u^h$**(x)** in element free Galerkin method is not equal to the nodal value of the function $u$**(x).** This is caused shape function of EFG method is not equal to kronecker delta. Namely:

$$\Phi_I(\mathbf{x}_J) \neq \delta_{IJ} \quad (36)$$

Thus should be imposed the essential boundary condition. A simple and efficient way for imposing essential boundary condition is penalty method. The essential boundary condition is:

$$\mathbf{u}=\hat{\mathbf{u}} \qquad \text{on} \qquad \boldsymbol{\Gamma}_{\mathbf{u}} \quad (37)$$

where **û** is the prescribed displacement on boundary $\boldsymbol{\Gamma}_{\mathbf{u}}$**.** The equation is obtained from weak form Galerkin with enforcing the essential boundary condition and using penalty method as [3]:

$$\int_\Omega \delta(\mathbf{Lu})^T c(\mathbf{Lu})d\Omega - \int_\Omega \delta\mathbf{u}^T.\, b\, d\Omega - \int_{\Gamma t}\delta\mathbf{u}^T .\, \tilde{t}\, d\Gamma - \delta\int_{\Gamma u}(.5)(u-\bar{u})^T.\alpha.(\, u-\bar{u})d\Gamma=0 \quad (38)$$

Applying mathematical calculation on (38), the final equilibrium equation is:

$$[\mathbf{K}+\mathbf{K}_{\alpha}]\ \mathbf{U}=\mathbf{P}+\mathbf{P}_{\alpha} \quad (39)$$

Where $\mathbf{K}_{\alpha}$ and $\mathbf{P}_{\alpha}$ are obtained for the essential boundary condition using penalty method, as:

$$\mathbf{K}_{\alpha}=\alpha\int_{\Gamma u}\mathbf{N}^{T}\mathbf{N}\ d\Gamma \quad (40)$$

$$\mathbf{P}_{\alpha}=\alpha\int_{\Gamma u}\mathbf{N}^{T}\bar{\mathbf{u}}\ d\Gamma \quad (41)$$

Where α is penalty factor. Ideally it is true to use infinite penalty factor. But if it is taken as infinite or too large, the numerical problems will be encountered. Thus the penalty factor is a number that the constraints be properly enforced. Usually it is equal to $10^{3}$E to $10^{13}$E and E is elasticity modulus.

**5- The Modified Penalty Method**

The enforcing of penalty method is necessary to satisfy essential boundary condition. The essential boundary condition is:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \Gamma_u \quad (42)$$

$\mathbf{K}_{\alpha}$ and $\mathbf{P}_{\alpha}$ in equation (39) properly should be determined. Liu [3] in their book (chapter 6) has said: "integration is performed along the essential boundary, and hence matrix $\mathbf{K}_{\alpha}$ will have entries only for the nodes near the essential boundary $\Gamma_u$, which are covered by the support domains of all the quadrature points on $\Gamma_u$."
If this statement is applied the results is not obtained exactly. Hughes [4] described the penalty method and obtained equilibrium equation with enforcing the essential boundary condition. **KU=P** is equilibrium equation without enforcing the essential boundary condition, and the essential boundary condition is:

$$d_Q = g \quad (43)$$

With enforcing the essential boundary condition is obtained Eq. (44):

$$(\mathbf{K}+k1_Q1_Q^{T})\mathbf{U} = \mathbf{P}+kg\mathbf{1}_Q \quad (44)$$

Where

$$\mathbf{1}_Q^{T}= [0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0] \quad (45)$$

and $k$ is penalty factor. Thus it is necessary to apply the penalty method and rewriting $\mathbf{K}_{\alpha}$ in the form of:

$$\mathbf{K}_{\alpha}=\boldsymbol{\alpha} \sum_{I=1}^{n} \mathbf{N}_I^{T}\mathbf{N}_I \quad (46)$$

Where $\mathbf{N}_I$ is obtained only for the nodes of boundary and is given the displacement, n is number of nodes of boundary with known displacement.

**6- Numerical Example:**
In this example a cantilever beam is studied (Fig. (1)). The displacement is calculated. The displacement between exact solution and present method is compared. The exact solution is given in Timoshenko and Goodier [5]:

$$u_x = -\frac{p}{6\ddot{E}I}\left(y-\frac{D}{2}\right)\left[3x\left(2L-x\right)+\left(2+\ddot{v}\right)y\left(y-D\right)\right] \quad (47)$$

$$u_y = \frac{p}{6\ddot{E}I}x^2\left(3L-x\right)+3\ddot{v}\left(L-x\right)\left(y-\frac{D}{2}\right)^2+\frac{4+5\ddot{v}}{4}D^2x \quad (48)$$

where

$$I=\frac{D^3}{12} \quad (49)$$

$$\ddot{E}=E,\ddot{v}=\nu \qquad \text{for plane stress} \quad (50)$$

$$\ddot{E}=\frac{E}{1-\nu^2},\ddot{v}=\frac{\nu}{1-\nu} \qquad \text{for plane strain} \quad (51)$$

The problem is solved for the plane strain case where p=-1000N, E=2e7Pa, D=4m, L=8m, ν=.3 and weight function is selected Cubicspline, number of quadrature point in each cell=6*6, number nodes=60(12*5), number cell=44(11*4), influence domain=1.955, penalty parameter=1e5. To compare are prepared exact solution with present solution for all nodes Figs. (2) to (4). The displacements of nodes along y=3 and y=4 coincide with the nodes along y=1 and y=0 respectively.

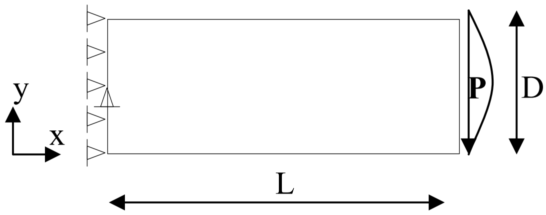


Fig (1): cantilever beam with force P distributed in a parabolic form at the end of the beam

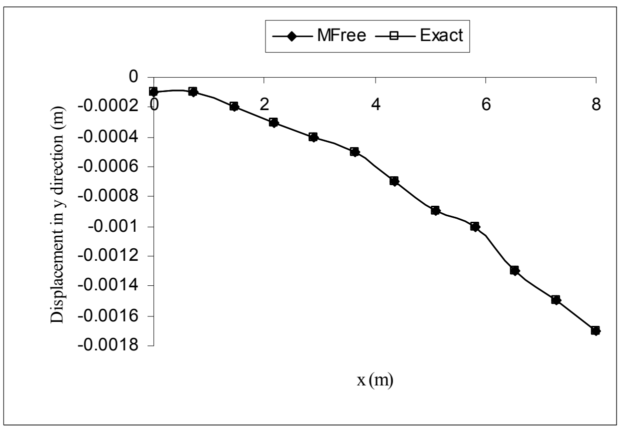


Fig (2): Displacement of the cantilever beam along y=0

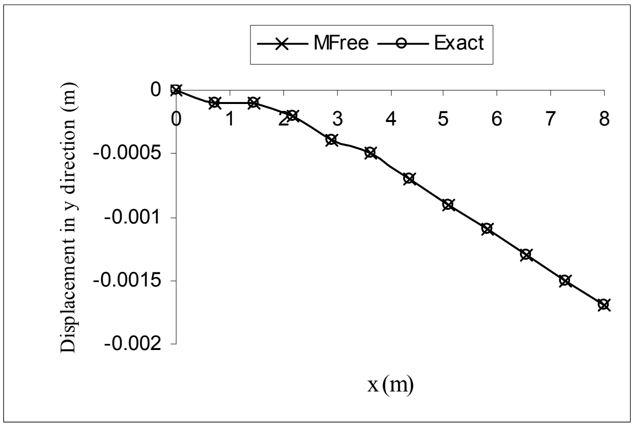


Fig (3): Displacement of the cantilever beam along y=1

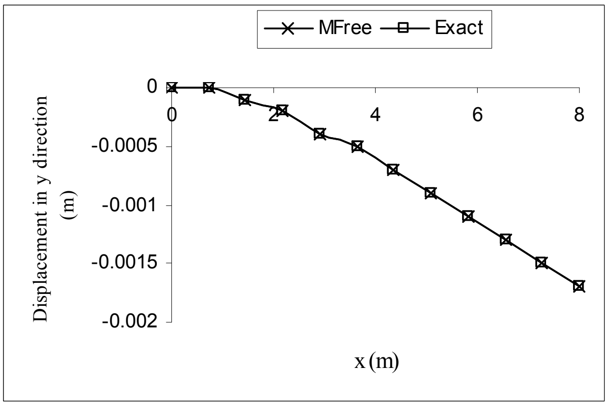


Fig (4): Displacement of the cantilever beam along y=2

## 7- Conclusion

The meshless methods are suitable and easy with respect to methods based on meshes. The based on meshes method for large deformations problems leads in reduction of speed and degradation of accuracy and complexity of the calculations. In this paper is used the Element Free Galerkin method product very accurate displacement results. This method can be used homogeneous and nonhomogeneous problems. The modified penalty method is used enforcing the essential boundary condition that is invention of this paper and obtained good results. To using the element free Galerkin method for discontinuous problems in the elastic state and the elsto-plastic state is used. The compare of exact solution and EFG method in example is shown efficiency and exact of EFG method.

## 8- References

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